Maximizing profit

Consider a market participant firm that operates under perfect competition. This firm sells its output at a constant market price p greater than zero, and incurs a cost w greater than zero for each unit of the single input it utilizes. The firm's production function is denoted by $f(x)$, with x being the amount of the variable input used. We aim to determine the input level x that will maximize the firm's profit. The production function is given by $f(x) = x^{1/4}$.

1. Write the profit function and show that is concave.
2. Find the critical points.
3. Find the profit, and show it is positive.

Solution

1. The profit Π of a competitive firm as a function of input x can be modeled by the expression

$$\Pi(x) = px^{1/4} - wx$$

Given a function f which is sufficiently smooth, specifically being twice-differentiable when $x > 0$, we examine its properties. The first derivative of the function, $f'(x)$, is calculated as $(1/4)x^{-3/4}$, and the second derivative, $f''(x)$, is found to be $-(3/16)x^{-7/4}$. Since the second derivative is negative for all positive x, the function f exhibits concavity over the domain $x > 0$. Moreover, because the function is continuous, it retains its concavity throughout the closed domain starting from $x = 0$.

Consequently, the profit function of the firm, denoted as $pf(x) - wx$, which is a composition of the production function and cost, inherits this property of concavity as it represents a combination of two concave functions.

2. The critical points of this function are obtained by setting the first derivative equal to zero, yielding $p(1/4)x^{-3/4} - w = 0$. Solving for x gives us the only critical point

$$x = \left(\frac{p}{4w}\right)^{4/3}$$

3. When x assumes the value $\left(\frac{p}{4w}\right)^{4/3}$, the profit is maximized, which can be calculated as follows:

$$\Pi\left(\left(\frac{p}{4w}\right)^{4/3}\right) = p\left(\left(\frac{p}{4w}\right)^{4/3}\right)^{1/4} - w\left(\frac{p}{4w}\right)^{4/3} = p\left(\frac{p}{4w}\right)^{1/3} - w\left(\frac{p}{4w}\right)^{4/3}$$

$$\Pi\left(\left(\frac{p}{4w}\right)^{4/3}\right) = \frac{p^{4/3}}{(4w)^{1/3}} - (\frac{p}{4})^{4/3}\frac{1}{w^{1/3}} \approx \frac{p^{4/3}}{w^{1/3}}0.472 > 0$$